Universal behavior in an irreversible model with C_{3v} symmetry

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We analyze the critical behavior of a two-dimensional irreversible cellular automaton whose dynamic rules are invariant under the symmetry operations of the point group C_{3v}. We study the dynamical phase transition that takes place in the model and obtain the static and short-time critical exponents by the use of Monte Carlo simulations. Our results indicate that the present model is in the same universality class as the three-state Potts model. [S1063-651X(99)04310-X]

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I. INTRODUCTION

Universality in nonequilibrium phase transitions, that is, in phase transitions between nonequilibrium steady states of irreversible systems, have been amply studied in recent years. Since the universal behavior depends only on a few general properties and not on the details of a system, it is reasonable to expect that equilibrium and nonequilibrium models which exhibit the same symmetries fall into the same universality class. This statement, established by Grinstein *et al.* [1] for systems with *up-down* symmetry (which includes the well known Ising model), follows from the renormalization group in $d=4-\epsilon$. That is, they have shown that, near the upper critical dimension, the irreversible parts of the Langevin equation are irrelevant with respect to the stable dynamical fixed point of the standard kinetic Ising model. This irrelevance continues to hold in lower dimensions. In fact, the proposition has been numerically verified for a large number of models [2–10].

It has also been noticed that there is a universal behavior in the early stages of evolution of a dynamical system. As predicted by Janssen *et al.* [11], when a system having a relaxation dynamics is quenched from a temperature much larger than the critical temperature to the critical temperature there is an initial increase of the order parameter described by a universal power law with a dynamic exponent θ. This behavior is observed when the system is already in the macroscopic short-time regime. The short-time universality has been verified for spin systems as the dynamic Ising model and Potts model [12–19]. In [20–22], the short-time dynamics analysis was performed to microscopically irreversible models containing the same symmetries as the kinetic Ising model. The calculated dynamic exponents are in agreement with those of the kinetic Ising model universality class. We point out here that the proposition introduced by Grinstein *et al.* [1] could be extended to comprehend the short-time universality.

In this paper we propose an irreversible probabilistic cellular automaton which has the same symmetries as the three-state Potts model [23]. We present a numerical study of the static and dynamical critical properties of the model defined in a regular square lattice and we focus our attention mainly on the determination of its short-time behavior. Our results indicate that the critical exponents obtained here and those associated to the two-dimensional three-state Potts model are the same within numerical errors. We might say, therefore, that lattice models invariant under the symmetries of the group C_{3v}, irreversible or not, are in the same universality class of the (equilibrium) three-state Potts model. Although this statement has not been demonstrated, it would be possible to prove it by following a renormalization-group reasoning similar to the one used by Grinstein *et al.* [1].

We remark that in the present case of an irreversible system, as opposed to equilibrium models, the symmetry operations are not defined as operations acting on a Hamiltonian but as operations acting on the evolution operator, i.e., on the dynamical rules.

II. THE MODEL

Consider a regular lattice of N sites in which each site can be in three states. At each time step, the state of the lattice is updated simultaneously according to the following local rules.

(a) If in the neighborhood of a given site there is a majority of sites which are in one state, then, independently of the state of the site, it changes to the state of the majority with probability p. It changes to one of the two other states with probability $(1-p)/2$.

(b) If no state is in majority, then the site assumes either state with equal probability.

The state of the system can be represented by $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_N)$, where $\sigma_i=1$, 2, or 3. Let $P_{\ell}(\sigma)$ be the probability of state σ at time ℓ and $w_i(\sigma|\sigma')$ be the transition probability per site. The time evolution equation of $P_{\ell}(\sigma)$ is given by

$$P_{\ell+1}(\sigma)=\sum_{\sigma'} W(\sigma|\sigma')P_{\ell}(\sigma'), \tag{1}$$

where

$$W(\sigma|\sigma')=\prod_i w_i(\sigma_i|\sigma') \tag{2}$$

is the transition probability from state σ' to state σ, given that at the previous time step the system was in state σ'.

For the case of a square lattice, we write $w(\sigma_0|\sigma_1,\sigma_2,\sigma_3,\sigma_4)$, where the sites 1, 2, 3, and 4 are the first neighbors of site 0. According to the local rules of the model, we have

$$w_i(1|1111)=w_i(1|1112)=w_i(1|1113)=w_i(1|1123)=p \tag{3}$$

and

$$w_i(1|1122)=w_i(1|1133)=w_i(1|2233)=1/3. \tag{4}$$

The other rules are obtained by permutation of the neighboring sites and by cyclic permutation of the states.

The transition probability $W(\sigma|\sigma')$ is invariant under certain symmetry operations, that is, $W(R\sigma|R\sigma')=W(\sigma|\sigma')$, where R is a symmetry operation. For the present model the symmetry operations are those that act on all sites transforming each of them in the same manner. One of the symmetry operations is the rotation operation $1\rightarrow 2$, $2\rightarrow 3$, and $3\rightarrow 1$. Another is the operator $2\rightleftharpoons 3$ with state 1 fixed. If the three states are placed on the vertices of an equilateral triangle, they correspond, respectively, to a rotation operation by 120 degrees and a specular operation. These symmetry operations define then the point group C_{3v}.

We remark that the dynamical rules so defined have the same symmetries as the Hamiltonian of the three-state Potts model, although in the present case the model is not defined by a Hamiltonian.

III. CRITICAL POINT

The system evolves in time according to the local rules given by Eqs. (2)–(4) and eventually reaches a steady state that can be of two types: a disordered steady state, where there is an equal average number of sites in each one of the three Potts states; and an ordered steady state characterized by the predominance of sites in one of the Potts states.

A convenient way to analyze the present model is through the use of the variables

$$x_\alpha=\frac{1}{N}\left(\sum_{i=1}^{N}\delta(\sigma_i,\alpha)-\frac{N}{3}\right), \tag{5}$$

where α assumes the values 1, 2, and 3 and $\delta(x,y)$ is the Kronecker delta. The property

$$x_1+x_2+x_3=0$$

holds so that just two of them are independent.

It is useful also to introduce a set of homogeneous functions $I_n(x_1,x_2,x_3)$, of a given order n, that are invariant under the symmetry operations R defined above. There is just one independent second-order invariant given by

$$I_2=\frac{1}{3}(x_1^2+x_2^2+x_3^2). \tag{6}$$

The fourth-order invariant function is

$$I_4=\frac{1}{3}(x_1^4+x_2^4+x_3^4). \tag{7}$$

Again there is just one independent fourth-order invariant.

The quantities of interest are the order parameter m defined by

$$m=\langle\sqrt{I_2}\rangle, \tag{8}$$

the *susceptibility* χ by

$$\chi=N\{\langle I_2\rangle-m^2\}, \tag{9}$$

and the reduced fourth-order cumulant U by

$$U=1-\frac{\langle I_4\rangle}{3\langle I_2\rangle^2}. \tag{10}$$

According to the theory of finite-size scaling, these quantities obey the following scaling forms:

$$m_L(p)=L^{-\beta/\nu}\tilde{m}(\epsilon L^{1/\nu}), \tag{11}$$

$$\chi_L(p)=L^{\gamma/\nu}\tilde{\chi}(\epsilon L^{1/\nu}), \tag{12}$$

and

$$U_L(p)=\tilde{U}(\epsilon L^{1/\nu}), \tag{13}$$

where $\epsilon=p-p_c$ is the deviation of p from its critical value, and $\tilde{m}(x)$, $\tilde{\chi}(x)$, and $\tilde{U}(x)$ are universal functions. For an infinite system these scaling forms give the behavior $m\sim\epsilon^\beta$ and $\chi\sim\epsilon^{-\gamma}$. The exponent ν is associated to the correlation length which diverges as $\xi\sim\epsilon^{-\nu}$. The cumulant $U_L(p)$, defined in Eq. (10), is expected to attain according to Eq. (13) a universal value at the critical point, which does not depend on the lattice size.

The simulation of the model is performed by applying the local rules in a synchronized way. We have considered square lattices and periodic boundary condition. Each simulation started with a configuration generated at random and averaged over several simulations where taken to get the final results. After a transient, which depends on the size of the system and of the value of parameter p, the system attains a steady state. Our simulations show that the system exhibits a continuous phase transition with the ordered steady state ($m\neq 0$) occurring at high values of p. As p is decreased, the transition takes place at a critical value p_c, and the system becomes disordered ($m=0$) for p less than p_c.

Using the cumulant method [24], the critical value p_c was estimated. As we can see from Fig. 1, the curves of U_L versus p, for different values of L, intercept at the critical point p_c estimated to be $p_c=0.888\pm 0.002$. Using the scaling relations for the order parameter and susceptibility, we have estimated the ratios β/ν and γ/ν. We found $\beta/\nu=0.134\pm 0.005$ and $\gamma/\nu=1.74\pm 0.02$, which are in agreement with the results for the (equilibrium) Potts model.

IV. SHORT-TIME BEHAVIOR

The short-time dynamic scaling relations [11] predict that at the early stage of the evolution of a system, at criticality, there is an initial increase of the order parameter that obeys a power-law behavior with a universal exponent defined as

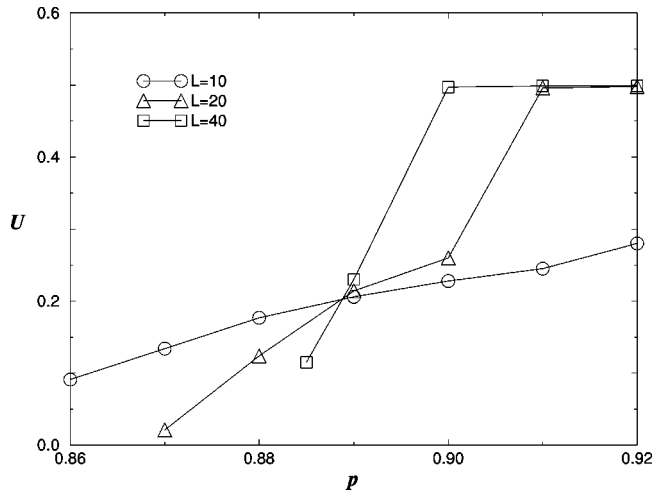

FIG. 1. The cumulant U_L versus p for square lattices with linear sizes $L=10$, 20, and 40.

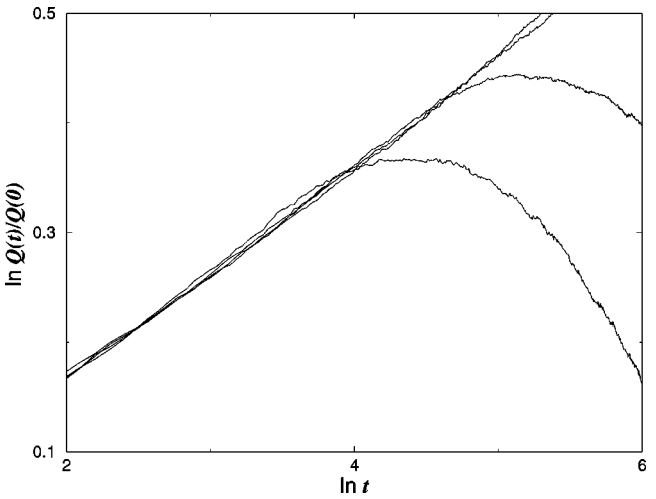

FIG. 2. Time evolution of $Q(t)/Q(0)$ for square lattices with linear sizes $L=9$, 18, 36, and 72 (from bottom to top).

$m(t)\sim m_0 t^{\theta}$, where m_0 is the initial magnetization and θ is a dynamical critical exponent. Numerically θ can be found from this relation by preparing the system in a disordered state with a very small magnetization. The exponent should be evaluated by taking the limit as m_0 goes to zero [13–18].

This exponent can also be evaluated [21] by considering the initial time evolution of the following correlation function:

$$Q(t)=\langle x_\alpha(t)x_\alpha(0)\rangle \tag{14}$$

with x_α defined in Eq. (5), and the average is over a distribution with zero correlation length and zero magnetization. The following power-law increase is expected:

$$Q(t)\sim t^{\theta}. \tag{15}$$

In the present work we calculate θ from this formula.

We also calculated the second moment of the order parameter, given by

$$M_2=\frac{3}{4}\langle I_2\rangle.$$

So, according to the short-time scaling relations, at the critical point, we may expect the following power-law behavior:

$$M_2(t)\sim t^{\zeta} \tag{16}$$

with

$$\zeta=\frac{1}{z}\left(d-\frac{2\beta}{\nu}\right), \tag{17}$$

where d is the dimension of the lattice and z is the dynamical exponent associated to the time correlation length.

We performed numerical simulations and analyzed the short-time behavior of the model. We used square lattices with sizes $L=9$, 18, 36, and 72. The total number of independent initial configurations was of the order of 10^5. Each initial configuration was generated by placing sites at state 1, 2, and 3 with equal probability and independent of each other. After that, we allow the system to evolve in time according to the local rules with the parameter p fixed at its critical value $p_c=0.888$ (evaluated according to the procedure described in the preceding section). We calculated the value of the dynamical quantities whose average gives $Q(t)$ and $M_2(t)$ at each time step. The same procedure was repeated for a number of initial configurations from which the averages were obtained at each time step.

Figure 2 shows the plot of the quantity $Q(t)$, defined in Eq. (15), against the time t, in a double-log scale. $Q(t)$ increases with time and has a power-law behavior that can be observed since the early times of the evolution even for small lattices. We see then that for the measurement of θ, finite-size effects are not important and the slope of straight lines fitted to the data points gives the exponent θ. We notice that for lattice sizes $L\geqslant 18$, the value of θ converges to a definite value, namely $\theta=0.093\pm 0.004$.

In Fig. 3 we show the quantity $M_2(t)$ versus time t. In this case there appears a nonuniversal behavior in the beginning times of the evolution. Discarding the first 20 initial time steps, we see that $M_2(t)$ presents a power-law behavior, and

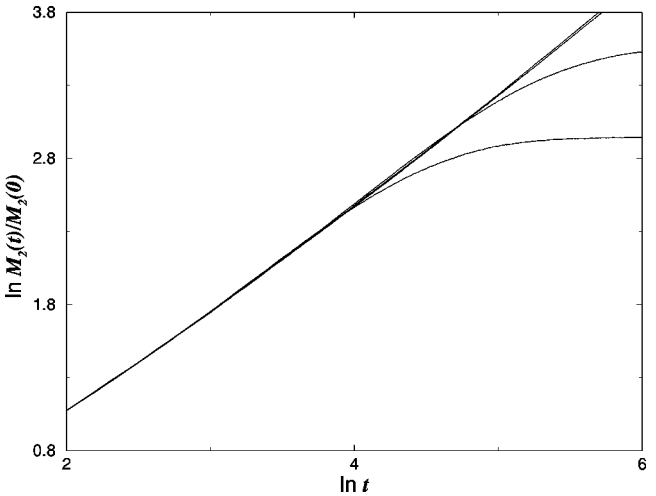

FIG. 3. Second moment $M_2(t)/M_2(0)$ as a function of time for square lattices with linear sizes $L=9$, 18, 36, and 72 (from bottom to top).

the slope of the straight lines fitted to the data points gives, in the plot of $\ln M_2(t)$ versus t, the exponent $\zeta = 0.76 \pm 0.01$. Using the value β/ν, obtained in the preceding section, on the relation $z = (d - 2\beta/\nu)/\zeta$, we obtain $z = 2.28 \pm 0.04$. The errors in z were calculated by propagating the errors in β/ν and ζ. This value of z is in agreement with the values of z, for the three-state Potts model, obtained in [25]. So, the short-time universal behavior of the present model is consistent with the short-time universal behavior of the kinetic three-state Potts model.

V. SUMMARY

We have investigated the critical behavior of a probabilistic cellular automaton that does not have microscopic reversibility and is defined by a dynamics that is invariant under the symmetry operations of the point group C_{3v}. The kinetic phase transition that takes place, as the external parameter p is varied, is a continuous phase transition from a disordered steady state at small values of p to an ordered steady state at high values of p. We introduced a set of homogeneous functions that are invariant under the symmetry operations of the group C_{3v}. From these functions we define the order parameter, the susceptibility, and the fourth-order cumulant. The critical point and the static as well as the short-time critical exponents were estimated by numerical simulations on regular square lattices of different sizes and using finite-size scaling theory. The values of the exponents are consistent with those of the two-dimensional three-state Potts model.

The numerical results confirm that irreversible models with dynamics with the symmetries of the group C_{3v} and defined in regular lattices are in the same universality class as the (equilibrium) three-state Potts model defined in the same regular lattices. Local irreversibility plays an irrelevant role in the critical behavior and is not a property that might change the universality class of a given model. For the case of Ising symmetry, such a principle has been demonstrated by showing the irrelevance of irreversible terms under the renormalization group near the upper critical dimension [1]. For other symmetries, a similar renormalization-group approach could be used to demonstrate such a general principle.

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